Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Student Number

2021 **YEAR 12**

Mathematics Extension 1

Trial Examination

Q	Marks
MC	/10
11	/12
12	/17
13	/16
14	/15
Total	**/70**

General Instructions:

- Reading time – 10 minutes
- Working time – 2 hours
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations
- No white-out may be used

Total Marks: 70

Section I - 10 marks

- Allow about 15 minutes for this section

Section II - 60 marks

Allow about 1 hour 45 minutes for this section

This assessment task constitutes 0% of the course.

Section I

10 marks
Allow about 15 minutes for this section

Use the multiple-choice sheet for Question 1–10

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1 What is the coefficient of x^9 in $(x+5)^{16}$?

(A) ${}^{16}P_7 5^6$

(B) ${}^{16}P_9 5^9$

(C) ${}^{16}C_7 5^6$

(D) ${}^{16}C_9 5^9$

2 What is the derivative of $\sin^{-1} 5x$?

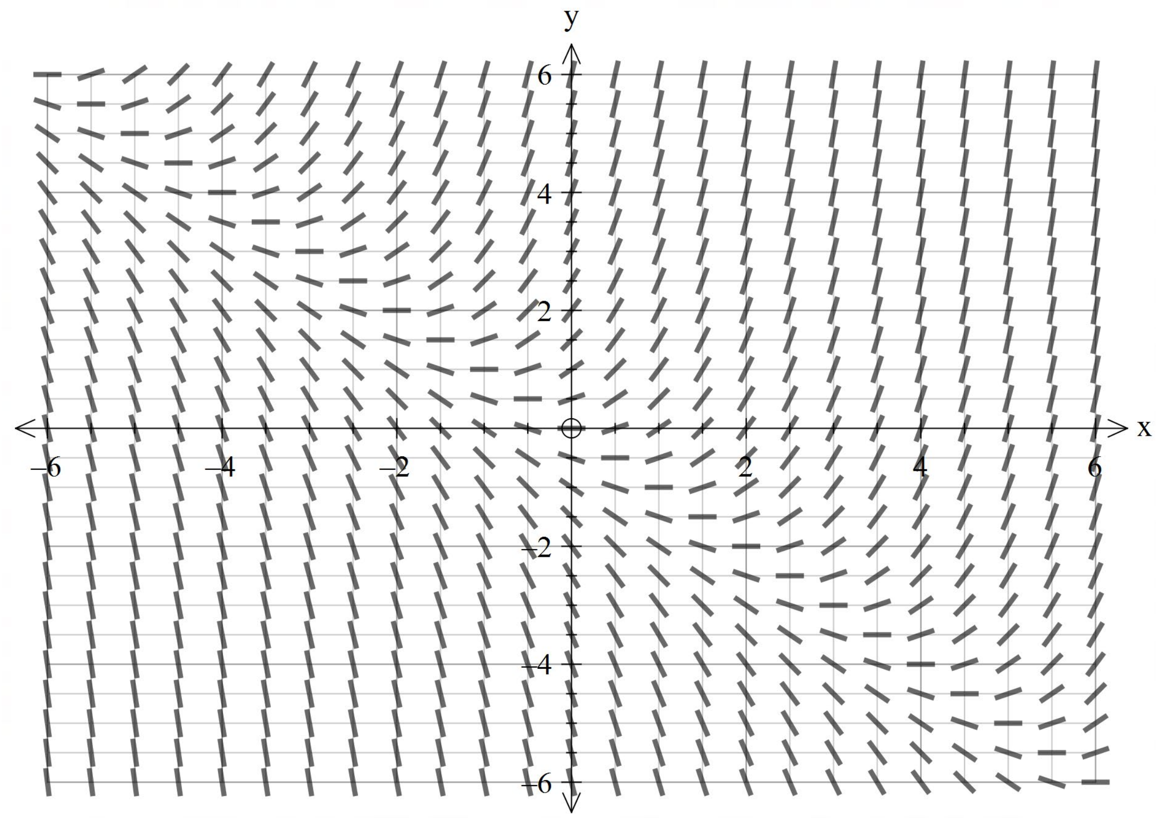
(A) $\dfrac{-5}{\sqrt{1-25x^2}}$

(B) $\dfrac{-1}{5\sqrt{1-25x^2}}$

(C) $\dfrac{1}{5\sqrt{1-25x^2}}$

(D) $\dfrac{5}{\sqrt{1-25x^2}}$

3



The slope field above is represented by which of the following differential equations.

(A) $\dfrac{dy}{dx} = \dfrac{2x}{y}$

(B) $\dfrac{dy}{dx} = x + y$

(C) $\dfrac{dy}{dx} = xe^{y}$

(D) $\dfrac{dy}{dx} = 1 + x$

4 Find the values of a and b given that $x+1$ and $x-2$ are factors of $ax^3 - 8x^2 + bx - 24$.

(A) $a = -20, b = 52$

(B) $a = -\frac{88}{7}, b = \frac{312}{7}$

(C) $a = \frac{88}{7}, b = -\frac{312}{7}$

(D) $a = 20, b = -52$

5 Which of the following is equivalent to the vector $-\boldsymbol{i} + \sqrt{3}\boldsymbol{j}$, in the form (r, θ), where r is the magnitude and θ is the angle with the positive x-aixs?

(A) $\left(-2, \frac{\pi}{3}\right)$

(B) $\left(2, \frac{\pi}{3}\right)$

(C) $\left(-2, \frac{2\pi}{3}\right)$

(D) $\left(2, \frac{2\pi}{3}\right)$

6 If $\sin\alpha = \frac{2}{3}$ and $\cos\alpha = \frac{\sqrt{5}}{3}$ evaluate $\cos 2\alpha$.

(A) $-\frac{1}{3}$

(B) $\frac{1}{9}$

(C) $\frac{4\sqrt{5}}{3}$

(D) $\frac{4\sqrt{5}}{9}$

7 Evaluate

$$^{n-1}C_{k-1} + {}^{n-1}C_k$$

(A) $\dfrac{n!}{(n-k)!}$

(B) $\dfrac{n!}{k!\,(n-k)!}$

(C) $\dfrac{n!}{(k-1)!\,\big(n-(k-1)\big)!}$

(D) $\dfrac{n!}{\big(n-(k-1)\big)!}$

8 The scalar orthogonal projection $\begin{pmatrix} p \\ q \end{pmatrix}$ on to the unit vector $\begin{pmatrix} t \\ u \end{pmatrix}$ is

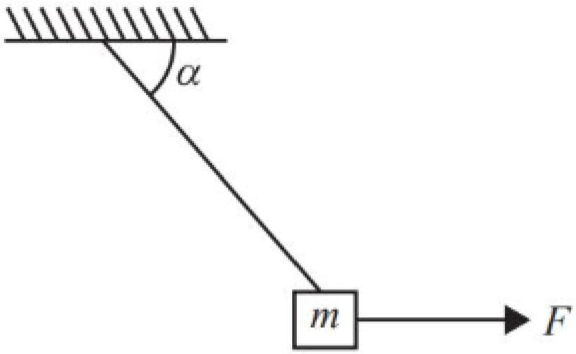
(A) $pt - qu$

(B) $pt + qu$

(C) $pu - qt$

(D) $pu + qt$

9 A particle of mass m kilograms hangs from a string that is attached to a fixed point. The particle is acted on by a horizontal force of magnitude F newtons/ The system is in equilibrium when the string makes an angle α to the horizontal, as shown in the diagram below. The tension in the string has magnitude T newtons.



The value of $\tan(\alpha)$ is

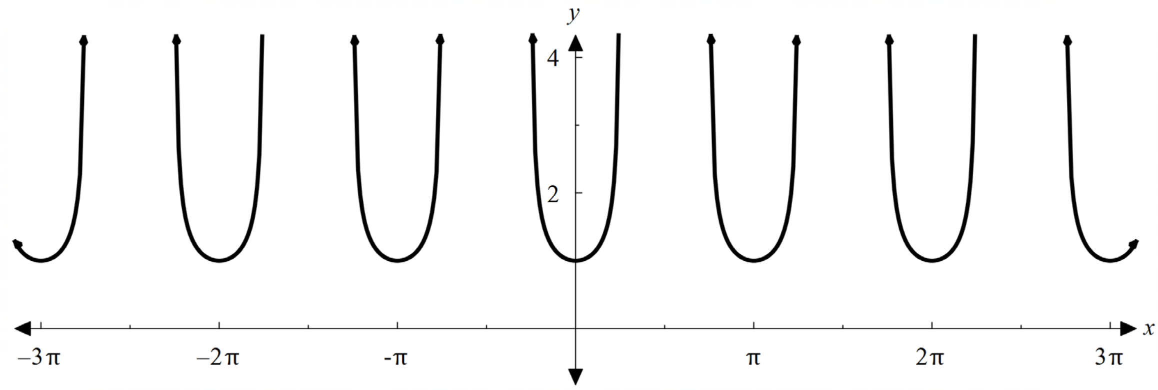
(A) $\dfrac{mg}{T}$

(B) $\dfrac{T}{mg}$

(C) $\dfrac{T}{F}$

(D) $\dfrac{mg}{F}$

10



Given the graph $y = \frac{1}{\sqrt{f(x)}}$ shown above, find $f(x)$.

(A) $f(x) = 2\sin(x)\cos(x)$

(B) $f(x) = 1 + \cos^2\left(x - \frac{\pi}{2}\right)$

(C) $f(x) = 1 - 2\sin^2(x)$

(D) $f(x) = \left|\tan\left(x + \frac{\pi}{2}\right)\right|$

End of Section I

Section II

In Questions 11 – 14, your response should include relevant mathematical reasoning and/or calculations.

Question 11 (12 marks)

(a) Solve for x. **2**

$$\frac{3x}{x-2} < 4$$

(b) (i) Show that $\sin 3x + \sin x = 2\sin 2x \cos x$. **1**

(ii) Hence, evaluate **2**

$$\int_0^{\frac{\pi}{4}} 2\sin 2x \cos x \, dx$$

(c) Solve the equation $x^3 + 2x^2 - 5x - 6 = 0$ given that one of its roots is equal to the sum of the other two roots. **2**

(d) Evaluate the following integral using the substitution $u = 1 - x^4$. **2**

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{2x^3}{\sqrt{1-x^4}} \, dx$$

(e) Use the t-formulae to find an expression for all possible solutions to the following equation **3**

$$3\cos x + 2\sin x = -2$$

Question 12 (17 marks)

(a) An online furniture store claims that 90% of all orders are shipped within 54 hours of a customer placing an order through the store's website. Ty ordered a total of 150 various pieces of furniture from the store for his company.

(i) According to the store's claim, 135 pieces of furniture should be delivered within 54 hours of Ty's order being placed through the store's website. **1**
What is the probability that this delivery will occur? Give your answer to three significant figures.

(ii) Show that the expected value and the standard deviation of the sampling proportion are 0.9 and 0.0245 respectively. **2**

(b) A scientist is studying a cell culture and finds the population, P, of the culture increases at a rate given by the equation $\frac{dP}{dt} = 1.7P$. **3**

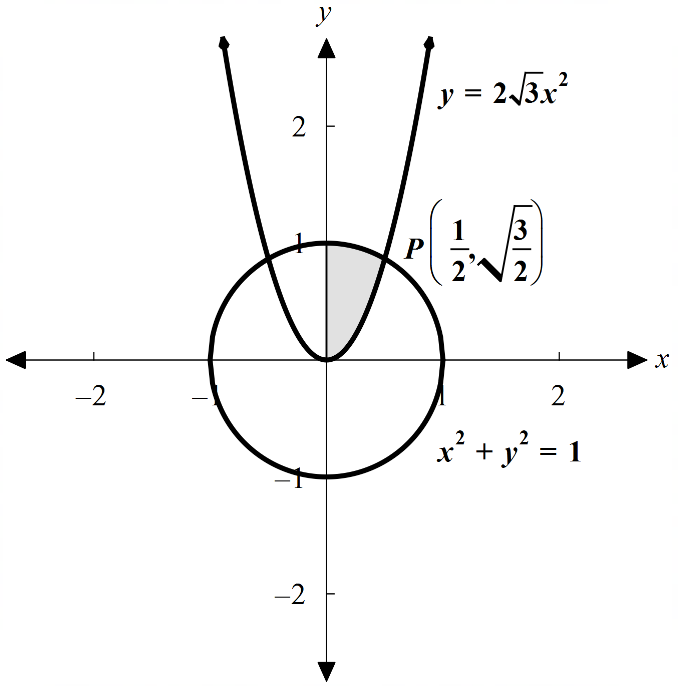
How long does it take for the population to increase to a level 2.5 times the initial population?

(c) In how many ways can all of the letters of the word PYTHAGORAS be arranged in a circle so that the consonants stay together? **2**

(d) Find the general solution of $\frac{dy}{dx} = e^{x-y}$. **2**

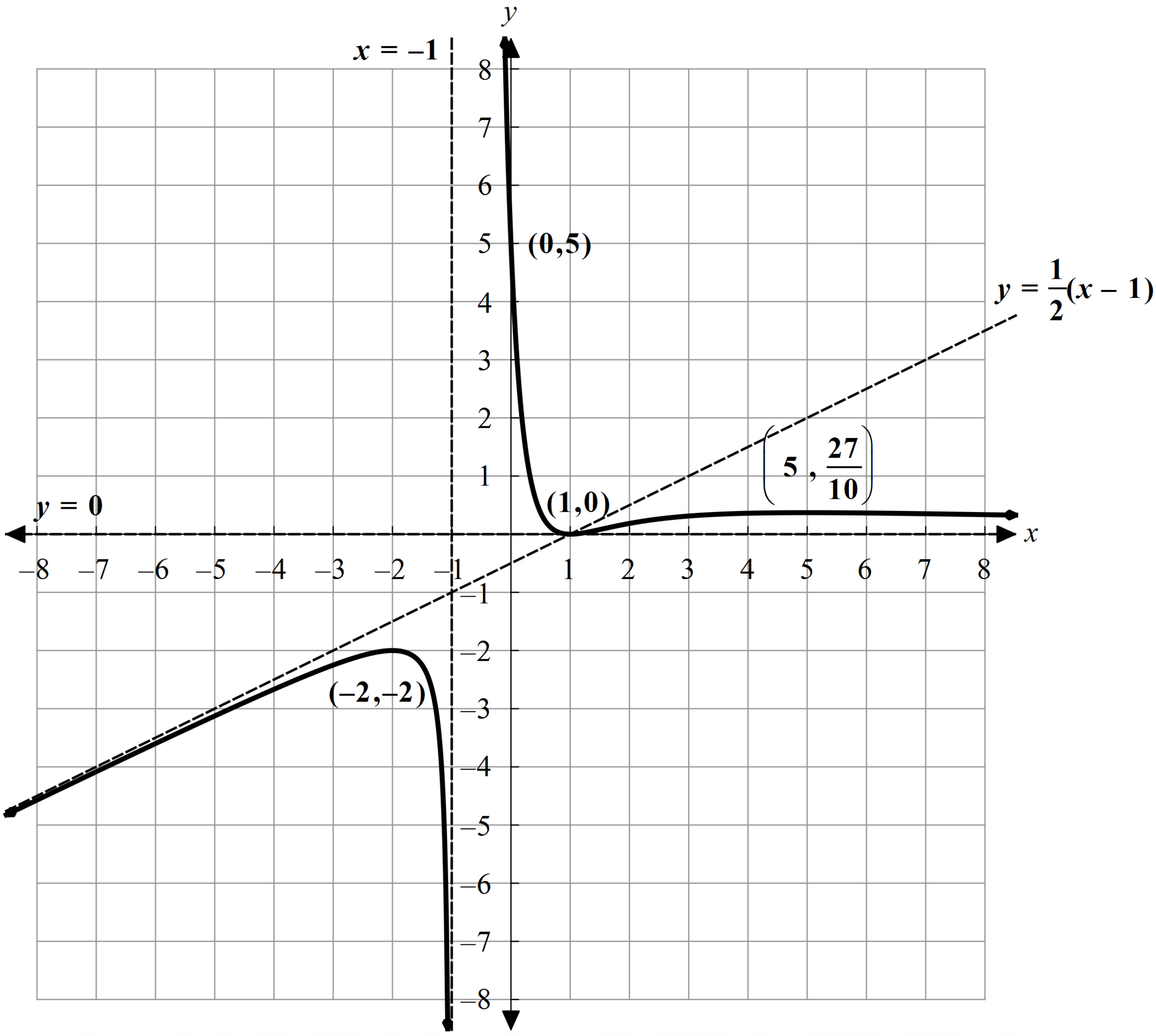
(e) Sketch the inverse $y = f^{-1}(x)$ given the function $f(x) = 2x^3 + 1$. **2**

(f) The diagram below shows the graph of $x^2 + y^2 = 1$ and $y = 2\sqrt{3}x^2$. The two graphs intersect at P. **2**



Find the volume of the solid of rotation where the shaded area is rotated about the x-axis.

(g) The graph of $y = f(x)$ has been sketched showing intercepts, turning points and asymptotes. Sketch the graph of $y = \frac{1}{f(x)}$ on the additional supplement provided. **3**



Question 13 (16 marks)

(a) A disease is spreading through a population at a rate proportional to the people who have it, and the people who do not. The rate at which it spreads is represented by the differential equation

$$\frac{dP}{dt} = \frac{1}{25}P\left(1 - \frac{P}{1000}\right)$$

(i) Simplify **1**

$$\frac{1}{P} + \frac{1}{K - P}$$

(ii) Hence, or otherwise, solve **3**

$$\frac{dP}{dt} = \frac{1}{25}P\left(1 - \frac{P}{1000}\right)$$

(b) Prove by Mathematical Induction that **3**

$$\log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \log\left(\frac{5}{4}\right) + \cdots + \log\left(\frac{n}{n-1}\right) = \log\left(\frac{n}{2}\right)$$

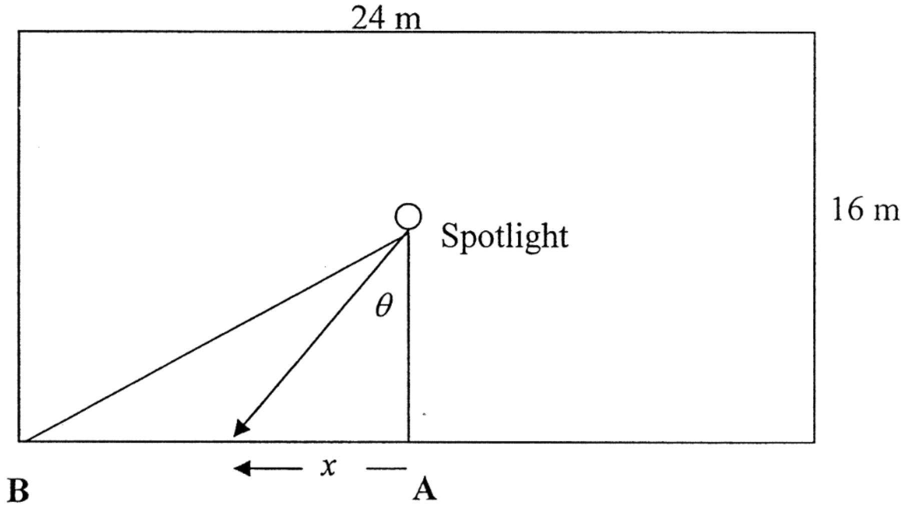
(d) Five distinct points are placed randomly within a square. Show that there must exist a pair of points that are at most $\frac{\sqrt{2}}{2}$ distance apart. **2**

(e) Expand $(1 + x)^{2n} + (1 - x)^{2n}$ and hence, evaluate **2**

$$1 + {}^{100}C_2 + {}^{100}C_4 + \cdots + {}^{100}C_{100}$$

Question 13 continues on next page

(f) A spotlight is in the centre of a rectangular nightclub which measures 24 m by 16 m. It is spinning at a rate of 20 rev/min. Its beam throws a spot which moves along the walls as it spins.



(i) Write the rate of rotation $\frac{d\theta}{dt}$ in radians/second. **1**

(ii) The spot moves along the wall from A to B at a velocity of $\frac{dx}{dt}$. **2**
Show that

$$\frac{dx}{dt} = \frac{16\pi}{3}\left(1 + \frac{x^2}{64}\right)$$

(iii) What is the difference in the velocities at which the spot appears to be moving at the points A, nearest to the light and B, furthest from the light? **2**

Question 14 (15 marks)

(a) (i) Show that $\cos 3x = 4\cos^3 x - 3\cos x$. **2**

(ii) Show that the solution of $\cos 3x - \sin 2x = 0$, for $0 < x < \frac{\pi}{2}$ is given by **2**

$$\sin x = \frac{\sqrt{5} - 1}{4}$$

(iii) Given that $x = \frac{\pi}{10}$ is a solution to $\cos 3x = \sin 2x$ and using the results obtained in parts (ii) prove **3**

$$\sin\frac{\pi}{5}\cos\frac{\pi}{10} = \frac{\sqrt{5}}{4}$$

(b) O is the bottom of a hill inclined at an angle of θ with the horizontal. A particle is projected from O with velocity V at an angle of elevation α with the hill. The particle lands at a point A on the hill.

(i) Derive the equations of motion of the particle along the OX and OY axes, where OX coincides with OA, and $OY \perp OX$, and both these axes lie in the projectile's plane. **2**

(ii) Hence, find the expressions for the time of flight and the range OA. **1**

(iii) Find the value of α that gives the maximum range. **2**

(iv) Given that the maximum range is obtained, prove that the initial velocity and the velocity at the landing point are perpendicular to each other. **2**

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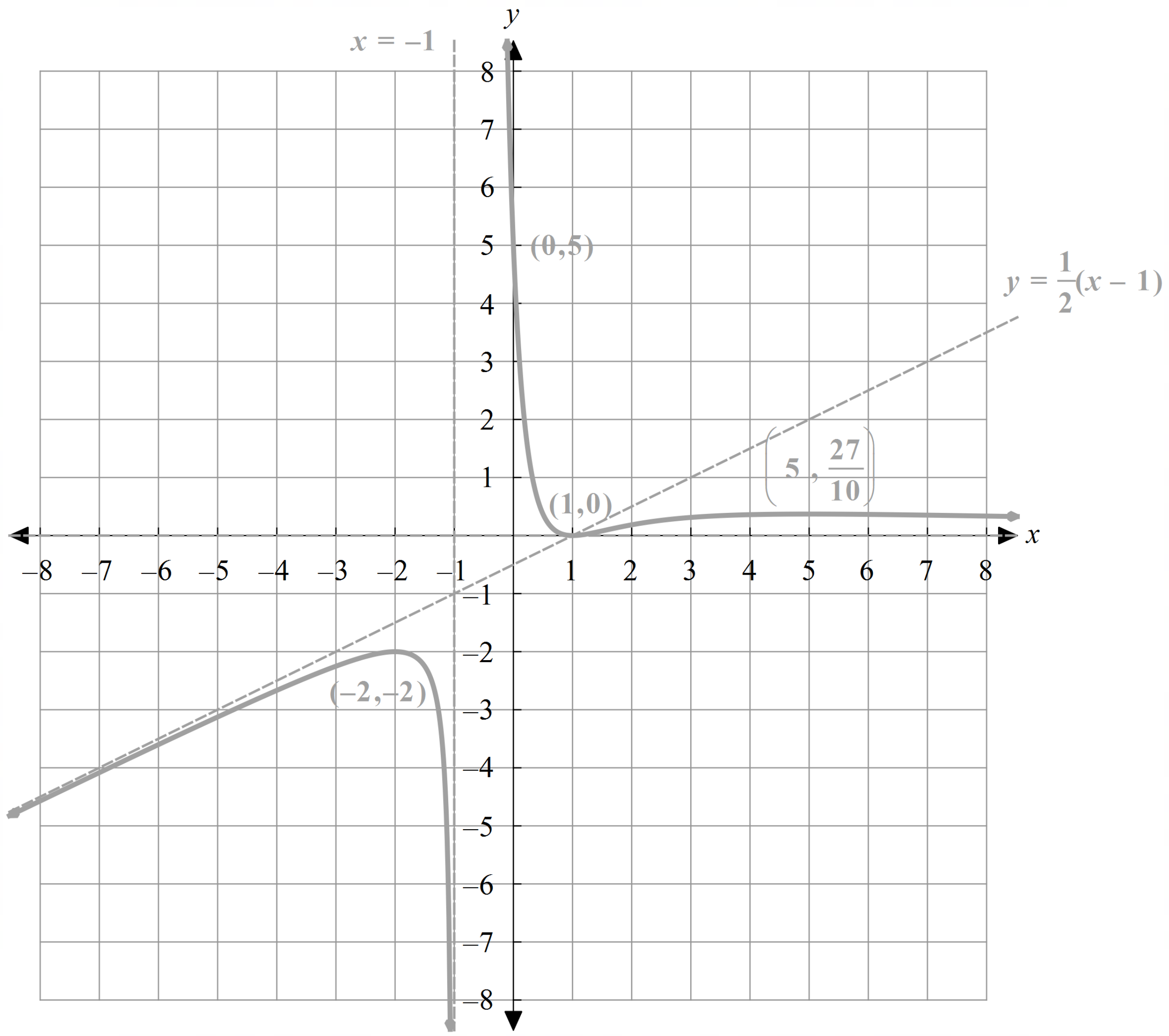
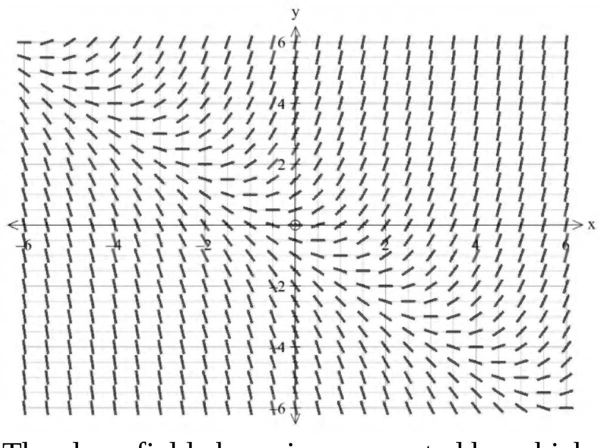
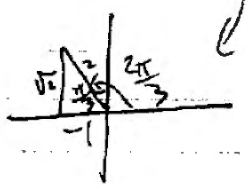
Question 12(g)

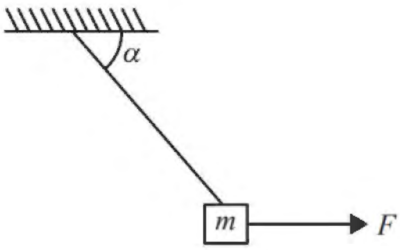
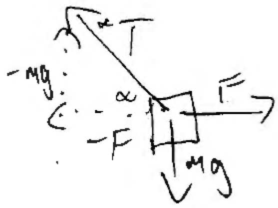
Given the graph of $y = f(x)$, sketch the graph of $y = \frac{1}{f(x)}$ below.



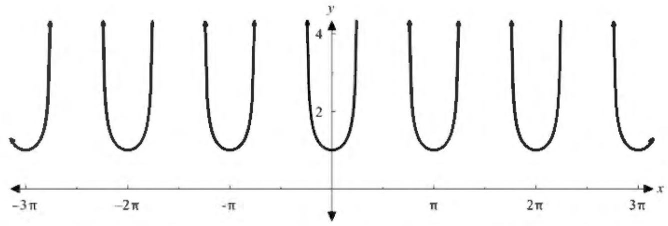
	Question	Marking guidelines
1	What is the coefficient of x^9 in $(x+5)^{16}$? 1. ${}^{16}C_9 (x)^9 (5)^{16-9}$ ${}^{16}C_9 5^7 x^9$ same as ${}^{16}C_7$ (C) ${}^{16}C_9 5^7$ (or none)	
2	What is the derivative of $\sin^{-1} 5x$? 2. $y = \sin^{-1}(5x)$ $\boxed{\frac{d}{dx}\sin^{-1}(f(x)) = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}}$ $y' = \frac{5}{\sqrt{1-25x^2}}$ D	
3	 The slope field above is represented by which of the following differential equations. B	

4	Find the values of a and b given that $x+1$ and $x-2$ are factors of $ax^3 - 8x^2 + bx - 24.$ 4. $P(-1) = -a - 8 - b - 24 = 0$ $a + b = -32.$ $P(2) = 8a - 32 + 2b - 24$ $8a + 2b = 56.$ $4a + b = 28$ (D)	
5	Which of the following is equivalent to the vector $-\boldsymbol{i} + \sqrt{3}\boldsymbol{j}$, in the form (r, θ), where r is the magnitude and θ is the angle with the positive x-axis? 5 $r = \sqrt{(-1)^2 + (\sqrt{3})^2}$ $= \sqrt{4}$ $= 2$ Also just obtuse.  $(2, \frac{2\pi}{3})$ (D)	

6	If $\sin\alpha = \frac{2}{3}$ and $\cos\alpha = \frac{\sqrt{5}}{3}$ evaluate $\cos 2\alpha$. 6. $\cos 2\alpha = \cos^2\alpha \mp \sin^2\alpha$ $= \left(\frac{\sqrt{5}}{3}\right)^2 - \left(\frac{2}{3}\right)^2$ $= \frac{5}{9} - \frac{4}{9}$. $= \frac{1}{4}$. (B)	
7	Evaluate $${}^{n-1}C_{k-1} + {}^{n-1}C_k$$ 7 ${}^{n-1}C_{k-1} + {}^{n-1}C_K$ $= \frac{(n-1)!}{(n-1-(k-1))!\,(k-1)!} + \frac{(n-1)!}{((n-1)-k)!\,k!}$ $= \frac{(n-1)!}{(n-1-k+1)(n-1-k)!\,(k-1)!} + \frac{(n-1)!}{(n-1-k)!\,k(k-1)!}$ $= \frac{k(n-1)! + (n-k)(n-1)!}{(n-k)!\,k!}$ $= \frac{\cancel{k(n-1)!} + n(n-1)! - \cancel{k(n-1)!}}{(n-k)!\,k!}$ $= \frac{n!}{(n-k)!\,k!}$ (B)	

8	The scalar orthogonal projection $\begin{pmatrix} p \\ q \end{pmatrix}$ on to the unit vector $\begin{pmatrix} t \\ u \end{pmatrix}$ is orthogonal unit vector to $\begin{pmatrix} t \\ u \end{pmatrix}$ 8 $\begin{pmatrix} p \\ q \end{pmatrix} \cdot \begin{pmatrix} u \\ -t \end{pmatrix} = pu - qt$ (C)	
9	A particle of mass m kilograms hangs from a string that is attached to a fixed point. The particle is acted on by a horizontal force of magnitude F newtons/ The system is in equilibrium when the string makes an angle α to the horizontal, as shown in the diagram below. The tension in the string has magnitude T newtons.  The value of $\tan(\alpha)$ is 9  $\tan\alpha = \frac{-mg}{-F}$ (D)	

10



Given the graph $y = \frac{1}{\sqrt{f(x)}}$ shown above, find $f(x)$.

10. $y = \frac{1}{\sqrt{f(x)}}$

$y(0) = y(\pi) = y(2\pi) = y(-\pi) = y(-2\pi)$ period of π

Even

$y = \frac{1}{\sqrt{\cos 2x}}$

$f(x) = \cos 2x$

$= 1 - 2\sin^2 x$

(C)

Question 11 (12 marks)

(a) Solve for x. **2**

$$\frac{3x}{x-2} < 4$$

$$\frac{3x}{x-2} < 4$$

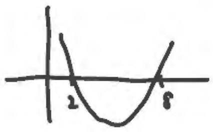
$3x(x-2) < 4(x-2)^2$

$4(x-2)^2 - 3x(x-2) > 0$

$(x-2)[4(x-2) - 3x] > 0$

$(x-2)(4x-8-3x) > 0$

$(x-2)(x-8) > 0$



$x < 2$ $x > 8$

1 mark for multiplying by the square of the denominator

1 mark for correctly solving the quadratic inequality

(b) (i) Show that $\sin 3x + \sin x = 2\sin 2x \cos x$. **1**

RTP: $\sin 3x + \sin x = 2\sin 2x \cos x$

LHS $= \sin 3x + \sin x$

$= 2\sin 2x \cos x$

$=$ RHS

$A + B = 3x$

$A - B = x$

$2A = 4x$

$A = 2x$

$2x + B = 3x$

$B = x$

1 mark for correctly applying sum to products formula

(ii) Hence, evaluate **2**

$$\int_0^{\frac{\pi}{4}} 2\sin 2x \cos x \, dx$$

$\int_0^{\pi/4} 2\sin 2x \cos \, dx$

$= \int_0^{\pi/4} \sin 3x + \sin x \, dx$ from part (i)

$= \left[-\frac{1}{3}\cos 3x - \cos x \right]_0^{\pi/4}$

$= -\frac{1}{3}\cos\frac{3\pi}{4} - \cos\frac{\pi}{4} - \left(-\frac{1}{3}\cos 0 - \cos 0\right)$

$= -\frac{1}{3} \times -\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{3} - 1\right)$

$= \frac{1}{3\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{4}{3}$

$= \frac{4 + 4\sqrt{2}}{3\sqrt{2}}$

1 mark for correctly integrating

1 mark for correctly evaluating the integral

(c) Solve the equation $x^3 + 2x^2 - 5x - 6 = 0$ given that one of its roots is equal to the sum of the other two roots. **2**

$x^3 + 2x^2 - 5x - 6 = 0$

Let α, β and $\alpha + \beta$ be the roots.

$\alpha + \beta + \alpha + \beta = -\frac{2}{1}$

$2(\alpha + \beta) = -2$

$\alpha + \beta = -1$

$\alpha = -1 - \beta \quad — \ ①$

$\alpha\beta(\alpha + \beta) = -\frac{(-6)}{1}$

$\alpha\beta(\alpha + \beta) = 6 \quad — \ ②$

① into ②

$\beta(-1-\beta)(-1-\beta+\beta) = 6$

$(-\beta - \beta^2)(-1) = 6$

$\beta^2 + \beta - 6 = 0$

$(\beta + 3)(\beta - 2) = 0$

$\beta = -3, \ \beta = 2 \quad — \ ③$

③ into ①

$\alpha = -1 - (-3) \qquad \alpha = -1 - 2$

$\alpha = 2 \qquad \alpha = -3$

The roots are 2, -3 and -1.

1 mark for correctly using sums and products of roots formula or any equivalent correct working.

1 mark for correctly finding all three roots.

(d) Evaluate the following integral using the substitution $u = 1 - x^4$. **2**

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{2x^3}{\sqrt{1-x^4}}\,dx$$

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{2x^3}{\sqrt{1-x^4}}\,dx$$

let $u = 1 - x^4$

$x^4 = 1 - u$

$\frac{du}{dx} = -4x^3$

$-\frac{du}{4} = x^3\,dx$

when $x = \frac{1}{\sqrt{2}}$

$u = 1 - \left(\frac{1}{\sqrt{2}}\right)^4$

$u = 1 - \frac{1}{4}$

$u = \frac{3}{4}$

when $x = 0$

$u = 1$

$$\int_0^{1/\sqrt{2}} \frac{2x^3}{\sqrt{1-x^4}}\,dx = \int_1^{3/4} \frac{-1}{2\sqrt{u}}\,du$$

$$= \int_{3/4}^{1} \frac{1}{2\sqrt{u}}\,du$$

$$= \left[\sqrt{u}\right]_{3/4}^{1}$$

$$= \sqrt{1} - \sqrt{\frac{3}{4}}$$

$$= 1 - \frac{\sqrt{3}}{2}$$

$$= \frac{2-\sqrt{3}}{2}$$

2 marks for correctly using the given substitution, including differentiating and changing the bounds.

1 mark penalty for any errors in working, including not changing the bounds, or incorrectly using the substitution.

(e) Use the t-formulae to find an expression for all possible solutions to the following equation **3**

$$3\cos x + 2\sin x = -2$$

$3\cos x + 2\sin x = -2$

let $t = \tan\frac{x}{2}$, then

$\frac{3(1-t^2)}{1+t^2} + \frac{2(2t)}{1+t^2} = -2$

$3 - 3t^2 + 4t = -2(1+t^2)$

$-3t^2 + 4t + 3 = -2 - 2t^2$

$-t^2 + 4t + 5 = 0$

$t^2 - 4t - 5 = 0$

$(t-5)(t+1) = 0$

$t = 5,\ t = -1$

solve, $5 = \tan\frac{x}{2}$

$\tan^{-1}(5) = \frac{x}{2}$

$2\tan^{-1}(5) = x$

$x = 2\tan^{-1}(5) \pm \frac{n\pi}{2}$

$-1 = \tan\frac{x}{2}$

$-\tan^{-1}(1) = \frac{x}{2}$

$\frac{x}{2} = -\frac{\pi}{4}$

$x = -\frac{\pi}{2} \pm \frac{n\pi}{2}$

test $x = \pi$

$3\cos(\pi) + 2\sin(\pi) = -3 + 0$
$= -3$

$x = 2\tan^{-1}(5) \pm 2n\pi$ or $x = -\frac{\pi}{2} \pm 2n\pi$

for $n \in \mathbb{Z}$

1 mark for correctly substituting the t-formulae

1 mark for correctly simplifying the expression in t and solving for the values of t.

1 mark for correctly solving for the value of x.

Question 12 (17 marks)

(a) An online furniture store claims that 90% of all orders are shipped within 54 hours of a customer placing an order through the store's website. Ty ordered a total of 150 various pieces of furniture from the store for his company.

(i) According to the store's claim, 135 pieces of furniture should be delivered within 54 hours of Ty's order being placed through the store's website. **1**
What is the probability that this delivery will occur? Give your answer to three significant figures.

$$P(X=135) = \binom{150}{135} 0.1^{15} \times 0.9^{135}$$
$$= 0.10797\ldots$$
$$\approx 0.108$$

1 mark for correct working and answer

(ii) Show that the expected value and the standard deviation of the sampling proportion are 0.9 and 0.0245 respectively. **2**

$$E(\hat{p}) = p$$
$$= 0.9$$
$$Var(\hat{p}) = \frac{0.9(1-0.9)}{150}$$
$$= \frac{3}{5000}$$
$$SD = \sqrt{\frac{3}{5000}}$$
$$= 0.0245$$

1 mark for showing that $E(\hat{p}) = 0.9$

1 mark for showing that the standard deviation is 0.0245

(b) A scientist is studying a cell culture and finds the population, P, of the culture increases at a rate given by the equation $\frac{dP}{dt} = 1.7P$. **3**

How long does it take for the population to increase to a level 2.5 times the initial population?

$$\frac{dP}{dt} = 1.7P$$
$$P = Ae^{kt}$$
$$\frac{dP}{dt} = k.Ae^{kt}$$
$$= k.P$$
$$\therefore k = 1.7$$
$$P = Ae^{1.7t}$$
$$2.5A = Ae^{1.7t}$$
$$2.5 = e^{1.7t}$$
$$\ln(2.5) = 1.7t$$
$$t = \frac{\ln(2.5)}{1.7}$$

1 mark for showing that $P = Ae^{kt}$ is a solution to the DE and use this to find the value of k.

1 mark for setting up the equation and solving for t.

(c) In how many ways can all of the letters of the word PYTHAGORAS be arranged in a circle so that the consonants stay together? **2**

PYTHAGORAS

7 consonants, 3 vowels.

"4" items to arrange - 3!

but 7! ways to arrange the consonants.

$\frac{3!7!}{2} = \frac{30240}{2}$

$= 15120$

1 mark for correctly identifying and arranging the 4 items in a circle.

1 mark for multiplying the number of these arrangements by the number of ways in which the consonants can be arranged.

(d) Find the general solution of $\frac{dy}{dx} = e^{x-y}$. **2**

$\frac{dy}{dx} = e^{x-y}$

$\frac{dy}{dx} = \frac{e^x}{e^y}$

$\int e^y \, dy = \int e^x \, dx$

$e^y = e^x + A$

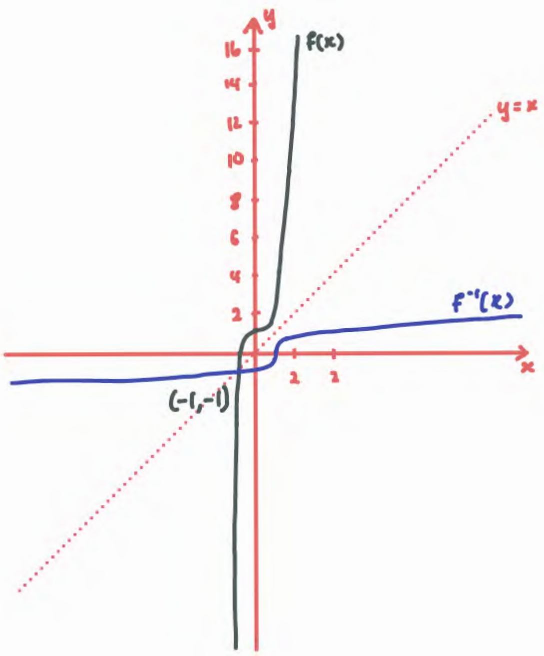
$y = \ln(e^x + A)$

1 mark for separating the DE

1 mark for integrating correctly and writing the solution with y as the subject.

(e) Sketch the inverse $y = f^{-1}(x)$ given the function $f(x) = 2x^3 + 1$. **2**

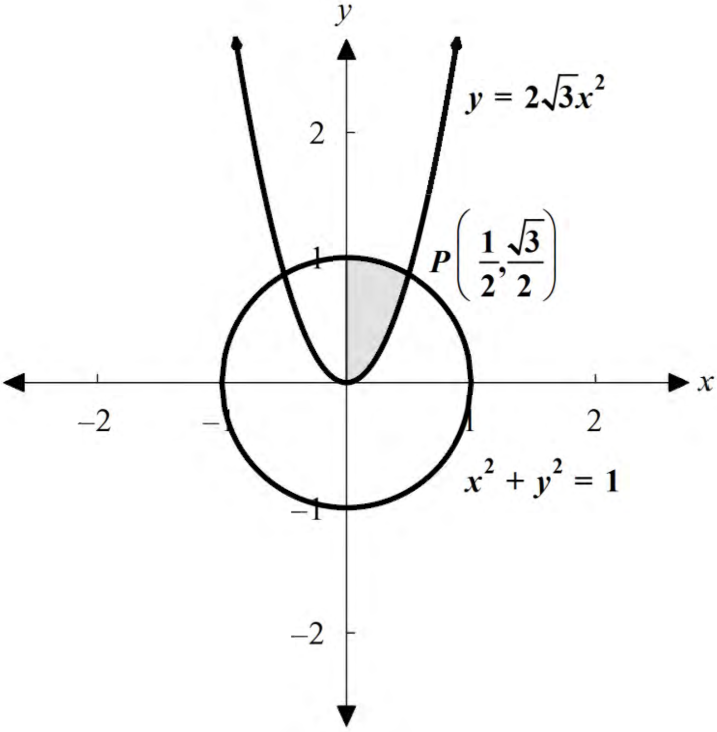
$y = 2x^3 + 1$



1 mark for correct x-intercept for $f^{-1}(x)$.

1 mark for showing the graph of $f^{-1}(x)$ passes through the point $(-1, -1)$

(f) The diagram below shows the graph of $x^2 + y^2 = 1$ and $y = 2\sqrt{3}x^2$. The two graphs intersect at P. **2**



Find the volume of the solid of rotation where the shaded area is rotated about the x-axis.

$x^2 + y^2 = 1$

$y^2 = 1 - x^2$

$y = 2\sqrt{3}x^2$

$y^2 = 12x^4$

$V = \pi\int_0^{\frac{1}{2}} 1 - x^2 - 12x^4 \, dx$

$= \pi\left[x - \frac{x^3}{3} - \frac{12x^5}{5}\right]_0^{\frac{1}{2}}$

$= \pi\left[\frac{1}{2} - \frac{(\frac{1}{2})^3}{3} - \frac{12(\frac{1}{2})^5}{5} - 0\right]$

$= \pi\left[\frac{1}{2} - \frac{1}{24} - \frac{3}{40}\right]$

$= \pi\left[\frac{60}{120} - \frac{5}{120} - \frac{9}{120}\right]$

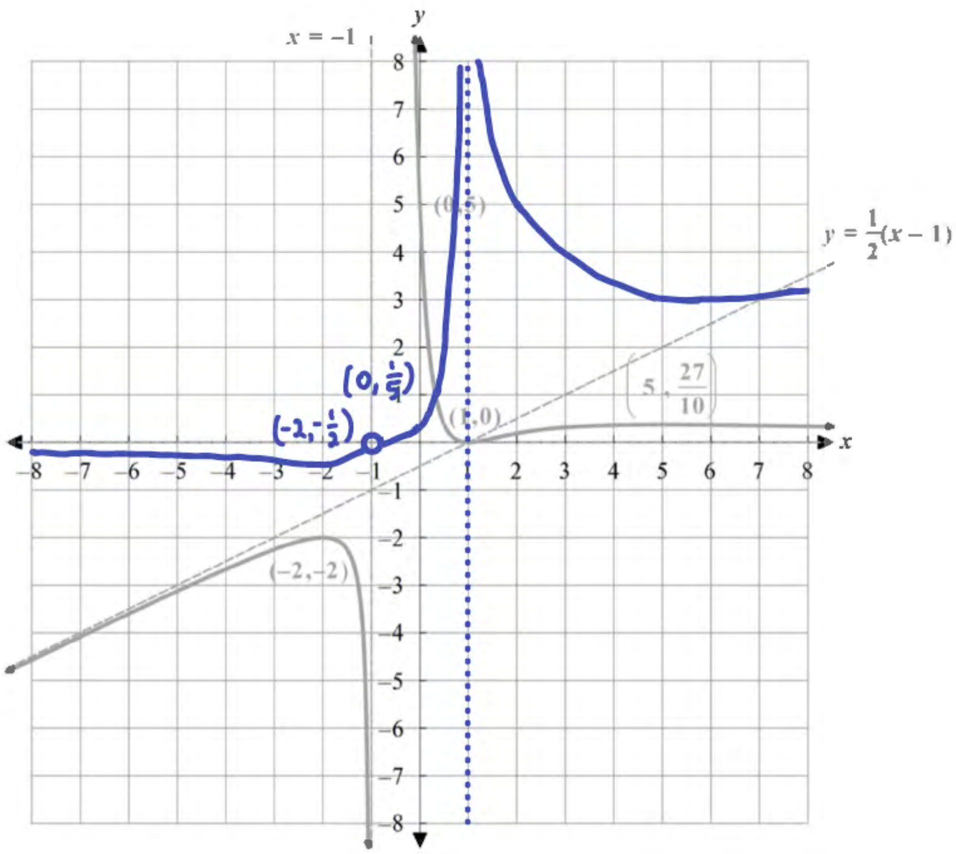
$= \frac{46\pi}{120}$

$= \frac{23\pi}{60}$

1 mark for subtracting the square of the lower curve from the square of the upper curve.

1 mark for correctly integrating and finding the volume of the solid of rotation.

(g) The graph of $y = f(x)$ has been sketched showing intercepts, turning points and asymptotes. Sketch the graph of $y = \frac{1}{f(x)}$ on the additional supplement provided. **3**



3 marks for correctly showing all critical features.

1 mark deducted for failing to show:

- Correct vertical asymptote at $x = 1$.
- Correct x-intercept
- Correctly showing that the graph of $\frac{1}{f(x)} \to 0^-$ as $f(x) \to -\infty$
- Correctly showing the y-intercept occurs at $\left(0, \frac{1}{5}\right)$

Question 13 (16 marks)

(a) A disease is spreading through a population at a rate proportional to the people who have it, and the people who do not. The rate at which it spreads is represented by the differential equation

$$\frac{dP}{dt} = \frac{1}{25}P\left(1 - \frac{P}{1000}\right)$$

(i) Simplify **1**

$$\frac{1}{P} + \frac{1}{K - P}$$

$$\frac{1}{P} + \frac{1}{K-P} = \frac{K - P + P}{P(K-P)} = \frac{K}{P(K-P)}$$

1 mark for correctly simplifying

(ii) Hence, or otherwise, solve **3**

$$\frac{dP}{dt} = \frac{1}{25}P\left(1 - \frac{P}{1000}\right)$$

$$\frac{dP}{dt} = \frac{1}{25}P\left(1 - \frac{P}{1000}\right)$$

$$\frac{dP}{dt} = \frac{P}{25} - \frac{P^2}{25.1000}$$

$$\frac{dP}{dt} = \frac{1000P - P^2}{25\,000}$$

$$\int \frac{1}{1000P - P^2}\,dP = \int \frac{1}{25000}\,dt$$

$$\int \frac{1000}{P(1000-P)}\,dP = \int \frac{1}{25}\,dt$$

$$\int \frac{1}{P} + \frac{1}{1000-P}\,dP = \int \frac{1}{25}\,dt$$

$$\ln(P) - \ln(1000-P) = \frac{t}{25} + c$$

$$\ln\left[\frac{P}{1000-P}\right] = \frac{t}{25} + c$$

$$\frac{P}{1000-P} = e^{t/25 + c}$$

$$P = Ae^{t/25}(1000-P) \quad \text{where } A = e^c$$

$$P = 1000Ae^{t/25} - P.Ae^{t/25}$$

$$P(1 + Ae^{t/25}) = 1000Ae^{t/25}$$

$$P = \frac{1000Ae^{t/25}}{1 + Ae^{t/25}}$$

$$P = \frac{1000}{Be^{-t/25} + 1} \quad \text{where } B = \frac{1}{A}$$

1 mark for correctly separating the DE.

1 mark for correctly integrating both sides

1 mark for simplifying the integral and writing the solution to the DE with P as the subject. No penalty applied for not dividing through by $Ae^{t/25000}$.

(b) Prove by Mathematical Induction that the following statement is true for all $n \geq 3$, **3**

$$\log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \log\left(\frac{5}{4}\right) + \cdots + \log\left(\frac{n}{n-1}\right) = \log\left(\frac{n}{2}\right)$$

$\log(\frac{3}{2}) + \log(\frac{4}{3}) + \log(\frac{5}{4}) + \cdots + \log(\frac{n}{n-1}) = \log(\frac{n}{2})$

Prove true for $n=3$,

LHS $= \log(\frac{3}{3-1})$

$= \log(\frac{3}{2})$

$=$ RHS

Assume true for $n=k$,

$\log(\frac{3}{2}) + \log(\frac{4}{3}) + \log(\frac{5}{4}) + \cdots + \log(\frac{k}{k-1}) = \log(\frac{k}{2})$

Prove true for $n=k+1$,

RTP: $\log(\frac{3}{2}) + \log(\frac{4}{3}) + \log(\frac{5}{4}) + \cdots + \log(\frac{k}{k-1}) + \log(\frac{k+1}{k}) = \log(\frac{k+1}{2})$

LHS $= \log(\frac{3}{2}) + \log(\frac{4}{3}) + \log(\frac{5}{4}) + \cdots + \log(\frac{k}{k-1}) + \log(\frac{k+1}{k})$

$= \log(\frac{k}{2}) + \log(\frac{k+1}{k})$, by assumption

$= \log(\frac{\cancel{k}}{2} \times \frac{k+1}{\cancel{k}})$

$= \log(\frac{k+1}{2})$

$=$ RHS

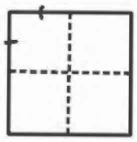
If the statement is true for $n=k$, then the the statement is true for $n=k+1$. Since the statement is true for $n=3$, then by mathematical induction, the statement is true for all $n \in \mathbb{Z}$ where $n \geq 3$

1 mark for correctly showing the statement is true for $n = 3$ and statement for assuming true for $n = k$.

1 mark for correctly proving statement is true for $n = k + 1$ and indicating where the assumption has been used in the proof.

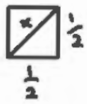
1 mark for final concluding statement.

(d) Five distinct points are placed randomly within a unit square. Show that there must exist a pair of points that are at most $\frac{\sqrt{2}}{2}$ units apart. **2**



4 pigeonholes and 5 pigeons.
Hence, there must be at least 2 points within one pigeon hole.

Each pigeonhole is a $\frac{1}{2} \times \frac{1}{2}$ unit square.



$$
\begin{aligned}
x &= \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} \\
&= \sqrt{\tfrac{2}{4}} \\
&= \tfrac{1}{\sqrt{2}} \\
&= \tfrac{\sqrt{2}}{2}
\end{aligned}
$$

1 mark for correctly identifying the pigeons and the pigeonholes.

1 mark for then showing that the two points are at most $\frac{\sqrt{2}}{2}$ units apart.

(e) Expand $(1 + x)^{2n} + (1 - x)^{2n}$ and hence, evaluate **2**

$$1 + {}^{100}C_2 + {}^{100}C_4 + \cdots + {}^{100}C_{100}$$

$(1+x)^{2n} + (1-x)^{2n}$

$1 + \binom{100}{2} + \binom{100}{4} + \dots + \binom{100}{100}$

$$
\begin{aligned}
(1+x)^{2n} + (1-x)^{2n} &= \binom{2n}{0}1^{2n}x^0 + \binom{2n}{1}x^1 + \binom{2n}{2}x^2 + \dots + \binom{2n}{2n}x^{2n} \\
&\quad + \binom{2n}{0}1^{2n}x^0 - \binom{2n}{1}x + \binom{2n}{2}x^2 - \dots + \binom{2n}{2n}x^{2n} \\
&= 1 + \cancel{\binom{2n}{1}x} + \binom{2n}{2}x^2 + \cancel{\binom{2n}{3}x^3} + \dots + \binom{2n}{2n}x^{2n} \\
&\quad + 1 - \cancel{\binom{2n}{1}x} + \binom{2n}{2}x^2 - \cancel{\binom{2n}{3}x^3} + \dots + \binom{2n}{2n}x^{2n} \\
&= 2\left[1 + \binom{2n}{2}x^2 + \binom{2n}{4}x^4 + \dots + \binom{2n}{2n}x^{2n}\right]
\end{aligned}
$$

So,

$$\tfrac{1}{2}\left[(1+x)^{2n} + (1-x)^{2n}\right] = 1 + \binom{2n}{2}x^2 + \binom{2n}{4}x^4 + \dots + \binom{2n}{2n}x^{2n}$$

when $n = 50$ and $x = 1$, then

$$\text{RHS} = \left[1 + \binom{100}{2} + \binom{100}{4} + \dots + \binom{100}{100}\right]$$

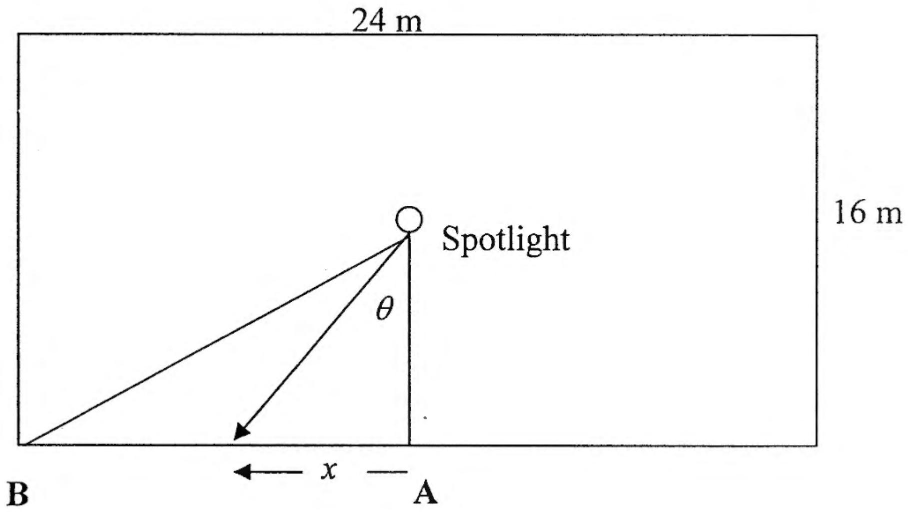
So,

$$
\begin{aligned}
\text{LHS} &= \tfrac{1}{2}\left[(1+1)^{100} + (1-1)^{100}\right] \\
&= \tfrac{1}{2} \cdot 2^{100} \\
&= 2^{99}
\end{aligned}
$$

1 mark for showing correct expansion.

1 mark for connecting expansion to the series and evaluating the sum of the series.

(f) A spotlight is in the centre of a rectangular nightclub which measures 24 m by 16 m. It is spinning at a rate of 20 rev/min. Its beam throws a spot which moves along the walls as it spins.



(i) Write the rate of rotation $\frac{d\theta}{dt}$ in radians/second. **1**

$$\frac{d\theta}{dt} = \frac{20 \times 2\pi}{60}$$
$$= \frac{2\pi}{3}$$

1 mark for correct answer

(ii) The spot moves along the wall from A to B at a velocity of $\frac{dx}{dt}$. **2**

Show that

$$\frac{dx}{dt} = \frac{16\pi}{3}\left(1 + \frac{x^2}{64}\right)$$

$$\tan\theta = \frac{x}{8}$$
$$\theta = \tan^{-1}\left(\frac{x}{8}\right)$$
$$\frac{d\theta}{dx} = \frac{\frac{1}{8}}{1+\left(\frac{x}{8}\right)^2}$$
$$= \frac{8}{64+x^2}$$
$$\frac{dx}{d\theta} = \frac{64}{8} + \frac{x^2}{8}$$
$$= 8 + \frac{x^2}{8}$$
$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$$
$$= \left(8 + \frac{x^2}{8}\right)\frac{2\pi}{3}$$
$$= \left(1 + \frac{x^2}{64}\right)\frac{16\pi}{3}$$
$$\frac{dx}{dt} = \frac{16\pi}{3}\left(1 + \frac{x^2}{64}\right)$$

1 mark for finding a derivative in terms of x and θ.
(Could differentiate $8\tan\theta$, and use the Pythagorean identities to convert from $\sec^2\theta$ to $\tan^2\theta$

1 mark for correct answer.

(iii) What is the difference in the velocities at which the spot appears to be moving at the points A, nearest to the light and B, furthest from the light? **2**

1 mark for finding A or B.

1 Mark for correct answer.

$$\frac{dx}{dt} = \frac{16\pi}{3}\left(1+\frac{x^2}{64}\right)$$

At A $x = 0$

$$\frac{dx}{dt} = \frac{16\pi}{3}$$

At B $x = 12$

$$\frac{dx}{dt} = \frac{16\pi}{3}\left(1+\frac{12^2}{8^2}\right)$$

$$= \frac{16\pi}{3}\left(1+\frac{\cancel{4^2}\times 3^2}{\cancel{4^2}\times 2^2}\right)$$

$$= \frac{16\pi}{3}\left(1+\frac{9}{4}\right)$$

$$= \frac{\overset{8}{\cancel{16}}\pi}{3}\left(\frac{13}{4^2}\right)$$

$$= \frac{104\pi}{3}$$

Difference between velocities

$$\frac{104\pi}{3} - \frac{16\pi}{3} = \frac{88\pi}{3} \text{ rad/second}$$

Question 14 (15 marks)

(a) (i) Show that $\cos 3x = 4\cos^3 x - 3\cos x$. **2**

$LHS = \cos 3x$

$= \cos 2x \cos x - \sin 2x \sin x$

$= (2\cos^2 x - 1)\cos x - 2\cos x \sin^2 x$

$= 2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x)$

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$

$= 4\cos^3 x - 3\cos x = RHS.$

2 marks for correct demonstration

1 mark for 1 application of a trigonometric identity

(ii) Show that the solution of $\cos 3x - \sin 2x = 0$, for $0 < x < \frac{\pi}{2}$ is given by **2**

$$\sin x = \frac{\sqrt{5}-1}{4}$$

$\cos 3x - \sin 2x = 0$

$4\cos^3 x - 3\cos x - 2\sin x \cos x = 0$

$\cos x(4\cos^2 x - 3 - 2\sin x) = 0$

$\cos x(4(1-\sin^2 x) - 2\sin x - 3) = 0$

$\cos x(4 - 3 - 2\sin x - 4\sin^2 x) = 0$

$-\cos x(4\sin^2 x - 2\sin x - 1) = 0$

$\sin x = \dfrac{-2 \pm \sqrt{4+16}}{8}$

$= \dfrac{-2 \pm \sqrt{20}}{8}$

$= \dfrac{-2 \pm 2\sqrt{5}}{8}$

$= \dfrac{-1 \pm \sqrt{5}}{4}$

$= \dfrac{-1+\sqrt{5}}{4}$ (discard negative as x is acute)

$= \dfrac{\sqrt{5}-1}{4}.$

1 mark – forms a quadratic in $\sin x$

1 mark – correct solution including the discarding of a negative.

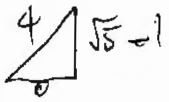
(iii) Given that $x = \frac{\pi}{10}$ is a solution to $\cos 3x = \sin 2x$ and using the results obtained in parts (ii) prove **3**

$$\sin\frac{\pi}{5}\cos\frac{\pi}{10} = \frac{\sqrt{5}}{4}$$

given that ~~sin~~ $x = \frac{\pi}{10}$ is a solution to $\cos 3x = \sin 2x$

$\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$

$\cos^2\frac{\pi}{10} = \frac{10+2\sqrt{5}}{16}$



$o^2 = 4^2 - (\sqrt{5}-1)^2$

$= 16 - (5 - 2\sqrt{5} + 1)$

$= 10 + 2\sqrt{5}$

$\text{LHS} = \sin\frac{\pi}{5}\cos\frac{\pi}{10}$

$= \left(2\sin\frac{\pi}{10}\cos\frac{\pi}{10}\right)\cos\frac{\pi}{10}$

$= 2\sin\frac{\pi}{10}\cos^2\frac{\pi}{10}$

$= \frac{2(\sqrt{5}-1)}{4} \times \frac{(10+2\sqrt{5})}{16}$

$= \frac{10\sqrt{5} + 10 - 10 - 2\sqrt{5}}{32}$

$= \frac{8\sqrt{5}}{32}$

$= \frac{\sqrt{5}}{4} = \text{RHS}$

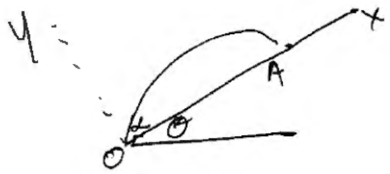
1 mark – Sets up right angled triangle, or equivalent.

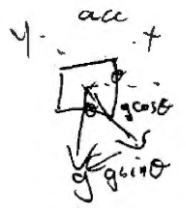
1 mark – Identifies $\sin\frac{\pi}{5}$ as $\sin\left(2\left(\frac{\pi}{10}\right)\right)$ and use this identity.

3 marks – correct solution.

(b) O is the bottom of a hill inclined at an angle of θ with the horizontal. A particle is projected from O with velocity V at an angle of elevation α with the hill. The particle lands at a point A on the hill.

(i) Derive the equations of motion of the particle along the OX and OY axes, where OX coincides with OA, and $OY \perp OX$, and both these axes lie in the projectile's plane. **2**





1 mark – split acceleration

2 marks – all equations of motion.

$\ddot{x} = -g\sin\theta$ $\qquad \ddot{y} = -g\cos\theta$

$\dot{x} = V_0\cos\alpha - gt\sin\theta$ $\qquad \dot{y} = V_0\sin\alpha - gt\cos\theta$

$x = V_0 t\cos\alpha - \frac{gt^2}{2}\sin\theta$ $\qquad y = V_0 t\sin\alpha - \frac{gt^2}{2}\cos\theta$

(ii) Hence, find the expressions for the time of flight and the range OA. **1**

let $y=0$ for max flight time ($g=10$)

$$y = V_0 t\sin\alpha - 5t^2\cos\theta = 0$$
$$t(V_0\sin\alpha - 5t\cos\theta) = 0$$
$$t = 0 \text{ or } V_0\sin\alpha = 5t\cos\theta$$
$$\frac{V_0\sin\alpha}{5\cos\theta} = t$$

sub $t = \frac{V_0\sin\alpha}{5\cos\theta}$ into x for range.

$$x = V_0\left(\frac{V_0\sin\alpha}{5\cos\theta}\right)\cos\alpha - 5\left(\frac{V_0^2\sin^2\alpha}{25\cos^2\theta}\right)\sin\theta$$

$$x = \frac{V_0^2\sin\alpha\cos\alpha}{5\cos\theta} - \frac{V_0^2\sin^2\alpha\sin\theta}{5\cos^2\theta}$$

$$= \frac{V_0^2\sin\alpha}{5\cos\theta}\left(\cos\alpha - \sin\alpha\tan\theta\right)$$

$$= \frac{V_0^2\sin\alpha}{5\cos^2\theta}\left(\cos\alpha\cos\theta - \sin\alpha\sin\theta\right)$$

$$= \frac{V_0^2\sin\alpha\cos(\alpha+\theta)}{5\cos^2\theta}$$

1 for correct expressions for flight and range.
(Not penalised for not using the compound angle formula for the range.)

(iii) Find the value of α that gives the maximum range. **2**

1 mark correct differentiation of x

1 mark for correct solution

(iii) $\frac{dx}{d\alpha} = \frac{V_0^2}{5\cos^2\theta}\left(\cos\alpha\cos(\alpha+\theta) - \sin\alpha\sin(\alpha+\theta)\right)$

$= \frac{V_0^2}{5\cos^2\theta}\left(\cos(2\alpha+\theta)\right)$

For stationary points let $\frac{dx}{d\alpha} = 0$

$= \frac{V_0^2}{5\cos^2\theta}\left(\cos(2\alpha+\theta)\right) = 0$

$\cos(2\alpha+\theta) = 0$

$2\alpha+\theta = \frac{\pi}{2}$ $\quad\left(\alpha < \frac{\pi}{2}\ \ \theta < \frac{\pi}{2} \therefore 2\alpha+\theta < \frac{3\pi}{2}\right)$

$\alpha = \frac{\pi}{4} - \frac{\theta}{2}$

$\frac{d^2x}{d\alpha^2} = \frac{V_0^2}{5\cos^2\theta}\left(-2\sin(2\alpha+\theta)\right)$

$\frac{d^2x}{d\alpha^2} = \frac{V_0^2}{5\cos^2\theta}\left(-2\sin(2\alpha+\theta)\right)$

sub $\alpha = \frac{\pi}{4} - \frac{\theta}{2}$.

$= \frac{V_0^2}{5\cos^2\theta}\left(-2\sin\left(2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+\theta\right)\right)$

$= \frac{V_0^2}{5\cos^2\theta}\left(-2\sin\left(\frac{\pi}{2}-\theta+\theta\right)\right)$

$= \frac{V_0^2}{5\cos^2\theta}(-2)$

$\therefore$ As $\frac{d^2x}{d\alpha^2} < 0$ & $\frac{dx}{d\alpha} = 0$ $\quad \alpha = \frac{\pi}{4} - \frac{\theta}{2}$

gives the maximum range.

(iv) Given that the maximum range is obtained, prove that the initial velocity and the velocity at the landing point are perpendicular to each other. **2**

1 mark for finding some expression for the velocities at the maximum range.

2 marks – shows that the velocities are perpendicular, either with the dot product, or by finding the gradient and showing that $m_1 m_2 = -1$

Initial velocity

$\dot{x}_0 = V_0 \cos\alpha \qquad \dot{y}_0 = V_0 \sin\alpha \qquad \frac{y}{x} = \frac{V_0 \sin\alpha}{V_0 \cos\alpha} = \tan\alpha$

Final velocity (at $t = \frac{V_0 \sin\alpha}{5\cos\theta}$)

$\dot{x}_f = V_0 \cos\alpha - 10\left(\frac{V_0 \sin\alpha}{5\cos\theta}\right)\sin\theta$

$= V_0 \cos\alpha - 2V_0 \sin\alpha \tan\theta$

$\dot{y}_f = V_0 \sin\alpha - 10\left(\frac{V_0 \sin\alpha}{5\cos\theta}\right)\cos\theta$

$= V_0 \sin\alpha - 2V_0 \sin\alpha$

$= -V_0 \sin\alpha$

$\alpha = \frac{\pi}{4} - \frac{\theta}{2}$

$\theta = \frac{\pi}{2} - 2\alpha$

sub $\theta = \frac{\pi}{2} - 2\alpha$ into $\dot{x}_f$

$\dot{x}_f = V_0 \cos\alpha - 2V_0 \sin\alpha \tan\left(\frac{\pi}{2} - 2\alpha\right)$

$= V_0 \cos\alpha - 2V_0 \sin\alpha \cot 2\alpha$

$= V_0 \cos\alpha - 2V_0 \sin\alpha \cot 2\alpha$.

$V_0 \cdot V_f = x_0 x_f + y_0 y_f$

$= V_0 \cos\alpha \left(V_0 \cos\alpha - 2V_0 \sin\alpha \cot 2\alpha\right)$

$+ V_0 \sin\alpha \left(-V_0 \sin\alpha\right)$

$= V_0^2 \cos^2\alpha - 2V_0^2 \sin\alpha \cos\alpha \cot 2\alpha$

$- V_0^2 \sin^2\alpha$

$= V_0^2\left(\cos^2\alpha - \sin^2\alpha\right) - V_0^2 \sin 2\alpha \cot 2\alpha$

$= V_0^2 \cos 2\alpha - V_0^2 \sin 2\alpha \frac{\cos 2\alpha}{\sin 2\alpha}$

$= 0$

$\therefore V_0$ & V_f are perpendicular vectors.